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Damping of weakly nonlinear shallow-water waves

By JOHN W. MILES

Institute of Geophysics and Planetary Physics,
University of California, La Jolla

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Boundary-layer damping of one-dimensional gravity waves of slowly varying amplitude $a(t)$, characteristic wavenumber k, and characteristic frequency ω in water of depth d and kinematic viscosity ν is calculated for $a \ll d$, $d \ll 1/k$ and $(2\nu/\omega)^{\frac{1}{2}} \ll d$. General results are given for the temporal evolution of the power spectral density determined by either a Fourier-integral (spatially aperiodic) or Fourier-series (spatially periodic) representation of the wave. Solitary and cnoidal waves are considered as examples. Keulegan's (1948) inverse-fourth-power decay for the solitary wave is recovered, and the numerical parameter therein is evaluated by reduction to a Riemann zeta function. A universal decay curve is obtained for the Stokes-scaled amplitude $S = a/k^2d^3$ of the cnoidal wave as a function of the boundary-layer-scaled time $(\nu\omega)^{\frac{1}{2}}t/d$; the result is both more flexible and more compact than that obtained by Isaacson (1976). The decay is within 5 % of that for a solitary wave (inverse fourth power) for $S > 2$ or that for an infinitesimal wave (exponential) for $S < 2$. An analytical approximation with a maximum error of less than 1 % is obtained by joining an asymptotic approximation for $S > 1$ to the exponential approximation for $S < 1$.

1. The dissipation equations

We consider viscous damping of one-dimensional gravity waves of amplitude $a(t)$, characteristic length $1/k$, and characteristic frequency ω in a liquid of depth d and kinematic viscosity ν on the assumptions that

$$a \ll d, \quad d \ll 1/k, \quad \delta \equiv (2\nu/\omega)^{\frac{1}{2}} \ll d, \tag{1.1a–c}$$

which individually imply weak nonlinearity, weak dispersion and weak dissipation (in a boundary layer or boundary layers of thickness δ) and jointly imply that $a(t)$ is slowly varying ($|\dot{a}| \ll \omega a$).

Dissipation of the elementary progressive wave

$$\eta(x,t) = a(t)\cos\{k(x-ct)\}, \quad \omega = kc, \quad c = (gd)^{\frac{1}{2}}, \tag{1.2a–c}$$

in a channel of breadth b and depth d is governed by (cf. Landau & Lifshitz 1959, §§24, 25; Miles 1967)

$$da^2/dt = -2\gamma a^2, \tag{1.3}$$

where

$$\gamma = (\nu\omega/8d^2)^{\frac{1}{2}}[1+(2d/b)+\mathscr{C}], \tag{1.4}$$

and $\mathscr{C}$ is a surface-contamination parameter that has a maximum possible value of 2 and may be approximated by 1 for water that has been allowed to stand for a few hours in a typical wave tank. The dependence of $\mathscr{C}$ on frequency and the surface parameters is known theoretically (Miles 1967); however, the required data are typically unavailable, and we therefore regard $\mathscr{C}$ as an empirical constant.

We generalize (1.2) by admitting weak nonlinearity and weak dispersion in accordance with the restrictions (1.1), such that either

$$\eta(x,t) = (2\pi)^{-1}\int_{-\infty}^{\infty} A(\kappa,t)\,e^{i\kappa \mathfrak{x}}\,d\kappa \tag{1.5a}$$

or

$$\eta(x,t) = \sum_{-\infty}^{\infty} A_n(t)\,e^{in\kappa \mathfrak{x}}, \tag{1.5b}$$

where A and A_n are slowly varying, complex amplitudes,

$$\mathfrak{x} = x - \int_0^t \mathfrak{c}\,dt, \tag{1.6}$$

and $\mathfrak{c}$ is a slowly varying phase speed that may differ from c by $O(a/d)$ but is independent of either κ or n [we could choose $\mathfrak{c} \equiv c$ without loss of generality, thereby absorbing all of the slow variation of phase into the complex amplitude, but the representations (1.5a, b) are more convenient]. The reality of η implies the complex-conjugate relations

$$A(-\kappa,t) = A^*(\kappa,t), \quad A_{-n}(t) = A_n^*(t), \tag{1.7a,b}$$

whilst conservation of mass implies $A_0 = 0$ if η is measured from the quiescent free-surface level. The power spectral density is either $|A|^2$ (continuous) or $|A_n|^2$ (discrete); accordingly, the required generalization of (1.3) is either

$$\frac{d}{dt}\int_0^{\infty} |A(\kappa,t)|^2 d\kappa = -2\alpha\int_0^{\infty} |\kappa d|^{\frac{1}{2}}|A(\kappa,t)|^2 d\kappa \tag{1.8a}$$

or

$$\frac{d}{dt}\sum_1^{\infty} |A_n(t)|^2 = -2\gamma\sum_1^{\infty} n^{\frac{1}{2}}|A_n(t)|^2, \tag{1.8b}$$

where

$$\alpha \equiv (kd)^{-\frac{1}{2}}\gamma = (\nu c/8d^3)^{\frac{1}{2}}[1+(2d/b)+\mathscr{C}]. \tag{1.9}$$

The approximations (1.8a, b), which follow directly from the definition of *power spectral density*, also may be derived from the equations of motion through either the method of averaging (over $\mathfrak{x}$) or the method of multiple scales (cf. Ott & Sudan 1970).† They remain equally valid if t is replaced by x/c, as may be more convenient for progressive-wave calculations.

We apply the preceding results to solitary and cnoidal waves in §§2 and 3 by invoking the inviscid calculation of $\eta(x,t)$ as a first approximation, in which a^2 (*qua* measure of the wave energy) is conserved, and then calculating the decay of $a(t)$ from either (1.8a) or (1.8b). This procedure neglects both the dissipative and dispersive effects of viscosity on wave form (as opposed to wave amplitude) and therefore requires (1.1c) to be replaced by the stronger restriction $\delta \ll k^2d^3$.

† Keulegan (1948) derives a result that may be identified with (1.8a), with $\mathscr{C} = 0$ therein, by invoking the Parseval and convolution theorems for Fourier integrals.

2. Solitary wave

The profile and phase speed of a solitary wave of amplitude a are given by (Lamb 1932, §252, after invoking $a \ll d$)

$$\eta(x,t) = a\,\mathrm{sech}^2\{(3a/4d^3)^{\frac{1}{2}}x\}, \quad c = c\{1+(a/2d)\}. \qquad (2.1\,a,b)$$

The Fourier transform of η is

$$A = \tfrac{4}{3}\pi d^3\kappa\,\mathrm{cosech}\,\{\pi(d^3/3a)^{\frac{1}{2}}\kappa\}. \qquad (2.2)$$

Substituting (2.2) into (1.8a) and integrating the resulting differential equation yields

$$a(t) = a_0[1+C(a_0/d)^{\frac{1}{4}}\alpha t]^{-4}, \qquad (2.3)$$

where (see appendix A for evaluation of integrals)

$$C = \frac{\int_0^\infty x^{\frac{5}{2}}\,\mathrm{cosech}^2\,x\,dx}{3^{\frac{3}{4}}\pi^{\frac{1}{2}}\int_0^\infty x^2\,\mathrm{cosech}^2\,x\,dx} = \frac{\Gamma(\frac{7}{2})\,\zeta(\frac{5}{2})}{3^{\frac{3}{4}}(2\pi)^{\frac{1}{2}}\Gamma(3)\,\zeta(2)} = 0{\cdot}2372, \qquad (2.4)$$

and ζ is the Riemann zeta function [$\zeta(2) = \frac{1}{6}\pi^2$, $\zeta(\frac{5}{2}) = 1{\cdot}34149$].

The result (2.3) merely confirms that given by Keulegan (1948), which is equivalent to (2.3) with $C = 0{\cdot}236$ [the discrepancy between this value and (2.4) is perhaps smaller than might have been expected, since Keulegan's calculation of C involved the numerical evaluation of a double infinite integral].

3. Cnoidal wave

The profile and phase speed of a cnoidal wave of amplitude a and phase speed c are given by Lamb (1932, §253)

$$\eta(x,t) = a\left[\mathrm{cn}^2\left\{\left(\frac{3a}{4d^3}\right)^{\frac{1}{2}}\frac{x}{m};m\right\} - \left(\frac{\mathscr{E}-m'^2}{m^2}\right)\right], \qquad (3.1\,a)$$

$$c = c\left[1-\left(\frac{3\mathscr{E}-2+m^2}{2m^2}\right)\left(\frac{a}{d}\right)\right], \quad \mathscr{E} = \frac{E(m)}{K(m)}, \qquad (3.1\,b,c)$$

where cn is a Jacobi elliptic function of modulus m, K and E are complete elliptic integrals of the first and second kinds, $m' = (1-m^2)^{\frac{1}{2}}$ is the complementary modulus, a and m are related by the periodicity equation

$$a/(k^2d^3) = (4/3\pi^2)\,m^2K^2(m) \equiv S(m), \qquad (3.2)$$

and S is a Stokes-scaled amplitude. The Fourier-series expansion of η (cf. Cayley 1895, §384) yields

$$A_n = \tfrac{4}{3}k^2d^3nq^n(1-q^{2n})^{-1} \quad (A_0 \equiv 0), \qquad (3.3)$$

where

$$q = \exp\{-\pi K(m')/K(m)\}. \qquad (3.4)$$

Substituting (3.3) into (1.8b) and integrating the resulting differential equation yields

$$\gamma t = T(S) - T(S_0), \qquad (3.5)$$

where

$$T(S) = \int_{q(S)}^{1} F(q)\,dq, \quad F(q) = \frac{\dfrac{d}{dq}\sum_{1}^{\infty} n^2 q^{2n}(1-q^{2n})^{-2}}{2\sum_{1}^{\infty} n^{5/2} q^{2n}(1-q^{2n})^{-2}}, \qquad (3.6a, b)$$

$q(S)$ is implicitly determined by (3.2) and (3.4), S_0 is the initial value of S, and $a(t)/a_0 = S/S_0$. The upper limit of integration in (3.6a) has been chosen to yield $T \downarrow 0$ for $S \uparrow \infty$ $(q \uparrow 1)$.

Letting $m \downarrow 0$ $(S \downarrow 0)$ in (3.2) and (3.4) and invoking (AS17.3.11) and (AS17.3.21) [the prefix AS refers to an entry in Abramowitz & Stegun (1964), wherein m is equivalent to m^2 herein] yields $q \to \frac{3}{16}S$, after which $q \downarrow 0$ in (3.6b) yields $F \to 1/q$. Separating out the corresponding logarithmic singularity in (3.6a) then yields

$$T(S) = T_1 - \ln S + O(S) \quad (S \downarrow 0), \qquad (3.7a)$$

where

$$T_1 = \ln\left(\tfrac{16}{3}\right) + \int_0^1 [F(q) - q^{-1}]\,dq. \qquad (3.7b)$$

Numerical integration yields $T_1 = 3{\cdot}939$. Higher approximations can be obtained by expanding $T(S) + \ln S$ in powers of S but prove to be numerically inefficient; e.g. the next term in the expansion (3.7a) is $\frac{9}{8}S$.

Letting $m \uparrow 1$ in (3.2) and invoking (AS17.3.14) yields $S \to (4/3\pi^2)\ln^2(4/m')$; letting $m \uparrow 1$ in (3.4) then yields

$$q \sim \exp\{-\pi(3S)^{-\frac{1}{2}}\}[1 + O(\exp\{-\pi(3S)^{\frac{1}{2}}\})] \quad (S \uparrow \infty), \qquad (3.8)$$

which is in error by less than 3 % (0·5 %) for $S > 1$ $(S > 1{\cdot}5)$. Invoking the Euler–Maclaurin approximation for the sums in (3.6b) and combining the resulting approximation to T with (3.8), we obtain the asymptotic expansion (see appendix B for details)

$$T \sim 4{\cdot}2164 S^{-\frac{1}{4}}[1 - 0{\cdot}1225 S^{-\frac{1}{2}} + 0{\cdot}0805 S^{-\frac{3}{4}} - 0{\cdot}0197 S^{-\frac{5}{4}} + O(S^{-\frac{3}{2}})]. \qquad (3.9)$$

The function $T(S)$ provides the entire family of solutions for a/a_0 with S_0 as the family parameter. Numerical values, determined by computer integration of (3.6), are plotted in figure 1. The approximation (3.7a), which is equivalent to the exponential decay of the linear approximation, is in error by less than 1 % for $S < 1$ and by less than 5 % for $S < 2$. The approximation (3.9) is in error by less than 0·2 % for $S > 1$; the dominant term therein, which is equivalent to the inverse-fourth-power decay of the solitary wave, is in error by less than 5 % for $S > 2$. Joining (3.9) to (3.7a) at $S = 1$ therefore yields an analytical approximation with a maximum error of less than 1 %.

The results obtained in this section are basically equivalent to those recently obtained by Isaacson (1976) but are both more compact and more flexible. In particular, Isaacson works with a numerical, rather than an analytical, representation of the Fourier coefficients of η; accordingly, his results are essentially numerical and have to be calculated for each value of S_0. He gives graphical results for $m_0^2 K^2(m_0) \equiv \frac{3}{4}\pi^2 S_0 = 20$ and 50, which agree with those in figure 1 herein.

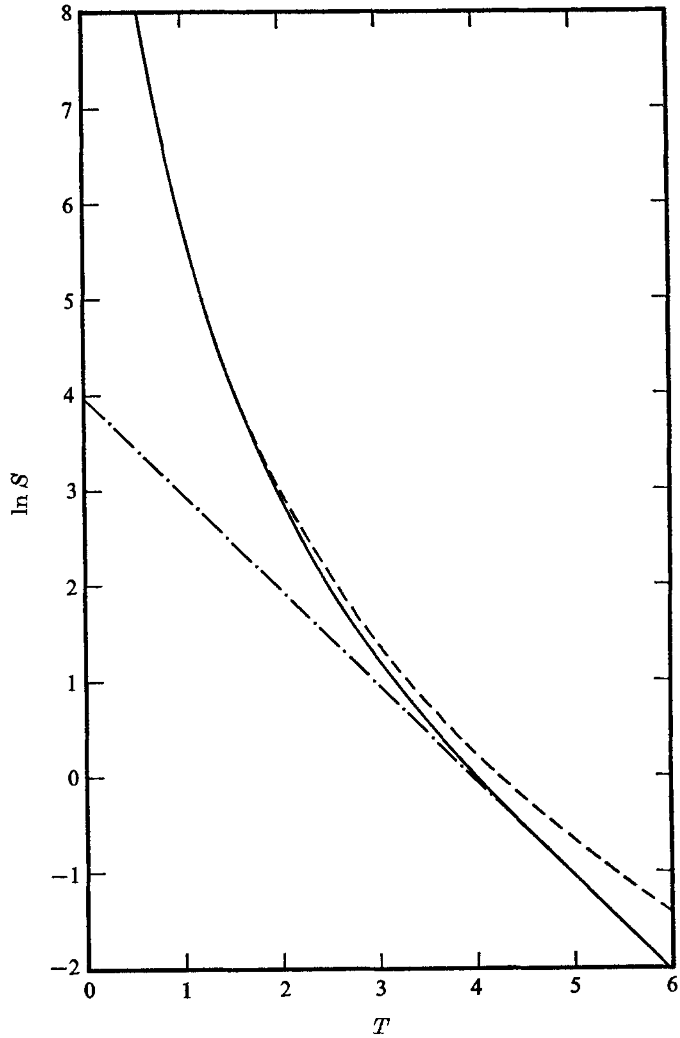

FIGURE 1. The universal decay function for cnoidal waves, as determined from (3.6a) by numerical integration. The particular result for a given initial value of S is obtained by measuring ln (a/a_0) vertically and $\gamma t = T - T_0$ horizontally from the point $\ln S = \ln S_0$, $T_0 = T(S_0)$. The asymptotes are $T \sim 3{\cdot}939 - \ln S$ (dot–dash curve) and $T \sim 4{\cdot}216 S^{-\frac{1}{4}}$ (dashed curve); see (3.7) and (3.9).

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Appendix A. Evaluation of integrals in (2.4)

The required integral is

$$I_s \equiv \int_0^\infty x^s \operatorname{cosech}^2 x \, dx. \tag{A 1}$$

Expressing cosech x in exponential form, introducing $t = 2x$, and integrating by parts yields

$$I_s = 4\int_0^\infty x^s e^{2x}(e^{2x}-1)^{-2}dx \tag{A 2a}$$

$$= 2^{1-s}\int_0^\infty t^s e^t(e^t-1)^{-2}dt \tag{A 2b}$$

$$= 2^{1-s}s\int_0^\infty t^{s-1}(e^t-1)^{-1}dt. \tag{A 2c}$$

Invoking the corresponding integral representation for the Riemann zeta function (AS 23.2.7) yields

$$I_s = 2^{1-s}\Gamma(s+1)\zeta(s). \tag{A 3}$$

Appendix B. Asymptotic evaluation of (3.6)

Introducing the change of variable

$$y = -\ln q \tag{B 1}$$

and

$$\mathscr{S}_m(y) \equiv \sum_{n=1}^{\infty} n^m q^{2n}(1-q^{2n})^{-2} \tag{B 2a}$$

$$= y^{-m}\sum_{n=1}^{\infty}(ny)^m e^{2ny}(e^{2ny}-1)^{-2} \equiv y^{-m}\sum_{n=1}^{\infty} f_m(ny) \tag{B 2b}$$

in (3.6) yields

$$T = -\frac{1}{2}\int_0^{-\ln q}\{\mathscr{S}_2'(y)/\mathscr{S}_{\frac{5}{2}}(y)\}dy. \tag{B 3}$$

Invoking the Euler–Maclaurin approximation (AS23.1.30, with $a = y$, $b = \infty$, $k = n-1$)

$$\sum_{n=1}^{\infty} f(ny) = y^{-1}\int_y^\infty f(u)\,du + \tfrac{1}{2}f(y) - \tfrac{1}{12}yf'(y) + \tfrac{1}{720}y^3 f'''(y) + \dots \tag{B 4}$$

yields

$$y^m\mathscr{S}_m = y^{-1}\int_0^\infty f_m(u)\,du - y^{-1}\int_0^y f_m(u)\,du + \tfrac{1}{2}f_m(y) - \tfrac{1}{12}yf_m'(y) + \tfrac{1}{720}y^3 f_m'''(y) + \dots. \tag{B 5}$$

Evaluating the infinite integral in (B 5) as in appendix A and expanding the finite integral in powers of y yields

$$\mathscr{S}_2 = \frac{\Gamma(3)\zeta(2)}{(2y)^3} - \frac{1}{8y^2} + O(y^4) \tag{B 6a}$$

and

$$\mathscr{S}_{\frac{5}{2}} = \frac{\Gamma(\frac{7}{2})\zeta(\frac{5}{2})}{(2y)^{\frac{7}{2}}} - \frac{399}{7680y^2} - \frac{1}{3584} - \frac{y^2}{28\,160} + O(y^4). \tag{B 6b}$$

Substituting (B 6) into (B 3) and invoking the asymptotic approximation (3.8) for q [note that the error in that approximation is exponentially small compared with the errors in (B 6)] yields

$$T = 1{\cdot}56538 \int_0^{\pi(3S)^{-\frac{1}{2}}} \left\{ \frac{1 - 0{\cdot}20264y + O(y^7)}{1 - 0{\cdot}13184y^{\frac{3}{2}} - 0{\cdot}00071y^{\frac{7}{2}} - 0{\cdot}00009y^{\frac{11}{2}}} \right\} \frac{dy}{y^{\frac{1}{2}}} \tag{B 7a}$$

$$= 1{\cdot}56538 \int_0^{\pi(3S)^{-\frac{1}{2}}} \{ y^{-\frac{1}{2}} - 0{\cdot}20264y^{\frac{1}{2}} + 0{\cdot}13184y - 0{\cdot}02672y^2 + O(y^{\frac{5}{2}}) \} \, dy, \tag{B 7b}$$

which yields (3.9).

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